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A new form of the eigenfunctions of the Heisenberg Hamiltonian in one dimension

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Abstract. The exact solution of the eigenvalue problem of a symmetric Heisenberg Hamiltonian in one dimension is investigated in a new form, in a spin-wave representation. This avoids the specific ordering of lattice points which characterizes the one-dimensional solution in configuration space, and may indicate a way to an extension to several dimensions. The familiar phase factors of the configuration space solution are determined independently of the dynamics of the equations, from the condition that there can be only one spin $\frac{1}{2}$ reversal on a lattice point, and from a zero-momentum condition resulting from the symmetry of the Hamiltonian. The new form of the solution reveals a number of new relationships, and a new method is presented to verify that these wavefunctions satisfy the wave equation.

1. Wave equations in configuration space and in a spin-wave representation

There has been a lively interest in recent years in the exact solutions of the Heisenberg Hamiltonian of a ferromagnet and their different extensions, especially because of their close relationship with simple exactly solvable models in statistical mechanics. The partition function of some two-dimensional lattice models is determined by the largest eigenvalue of a one-dimensional transfer matrix. As pointed out first by Lieb (1967) in connection with the residual entropy of ice, the eigenfunctions of the transfer matrix of a two-dimensional model of ice are given by the Bethe solution of a Heisenberg Hamiltonian with asymmetry parameter $\Delta = \frac{1}{2}$, and a knowledge of these eigenfunctions permits the determination of the eigenvalues. An extension of the method to ferroelectric and other models followed, but the considerable number of contributions to this subject will not be referred to here. Only Baxter's work (1972a, b) on the exact solution of the 8-vertex model in two dimensions should be mentioned. This is closely connected with the generalized XYZ form of the Heisenberg Hamiltonian in one dimension, and contains as a special case most of the presently known exactly solvable two-dimensional models.

On the other hand, especially at very low temperatures, two-dimensional ice is rather rare. Though recent experimental investigations produced measurements on one- and two-dimensional systems with considerable ingenuity, the main motivation of theoretical work on two-dimensional models is that they are easier to solve. It would be of some interest if at least a part of the results on these models could be extended to three dimensions.

Bethe gave the exact eigenfunctions of the Heisenberg Hamiltonian in one dimension (Bethe 1931) for a symmetric Hamiltonian. His work has been extended in several directions with the inclusion of an asymmetry parameter Δ (Orbach 1958, Walker 1959, Griffith 1964, Yang and Yang 1966b, c, d, des Cloizeaux and Gaudin 1966, Gaudin 1971, and other contributions), and much progress has been made in the understanding of the XYZ Hamiltonian with two asymmetry parameters in a more recent work by Baxter (to be published).

Bethe obtained his exact solution in configuration space, or lattice space, and was followed on the same lines by later investigators. The solution in configuration space is strongly dependent on a definite ordering $j_1 < j_2 < \ldots < j_n$ of the lattice points, which defines a domain, on the borders of which the wavefunction has discontinuities. This ordering of the points of lattice space gives an appearance to the solutions which is strongly one dimensional.

It is known, on the other hand (see for instance Yang and Yang 1966a), that in spite of all the important differences, the Heisenberg model of a ferromagnet has many similar features in one, two and three dimensions. These similarities should show up in some way in the form of the solutions. The single-spin excitations are known exactly in one, two and three dimensions and describe spin waves. Most approximation methods in calculating properties of a Heisenberg ferromagnet in three dimensions refer in some form to these spin-wave states. One can introduce a spin-wave representation which corresponds to a Fourier transformation of the configuration-space description and, if one applies such a Fourier transformation to Bethe's one-dimensional solution, the special ordering features of lattice space disappear. The jumps in the wavefunction at the borders of the ordered regions are also transformed away, corresponding to the fact that discontinuous functions may have continuous Fourier transforms.

The new form of the one-dimensional wavefunction investigated here will be essentially the Fourier transform of Bethe's solution. In establishing its properties and an independent method of verification, a first step may have been achieved towards an extension of the exact solution to several dimensions.

The asymmetry parameter Δ will be taken to be unity in the present paper. It will be seen that the extension to a general Δ is immediate. As in configuration space, the form (9*a*–*d*) of the eigenfunctions is independent of the asymmetry parameter, which influences only the expression of the phase factors. On the other hand, the greater symmetry of the Hamiltonian makes it possible to exploit the special role of zero-momentum states, and to present some new features which are clearest in this case.

The Hamiltonian of the symmetric Heisenberg ring with spin $\frac{1}{2}$ nearest-neighbour interactions will be chosen as

$$\mathscr{H} = \tfrac{1}{4} \sum_{j=1}^{N} (1 - \sigma_j^x \sigma_{j+1}^x - \sigma_j^y \sigma_{j+1}^y - \sigma_j^z \sigma_{j+1}^z) \tag{1a}$$

with a coupling constant $J = 1$, and with periodic boundary conditions

$$\sigma_{j+N}^x = \sigma_j^x, \qquad \sigma_{j+N}^y = \sigma_j^y, \qquad \sigma_{j+N}^z = \sigma_j^z. \tag{1b}$$

The Pauli matrices σ_j^x, σ_j^y, σ_j^z have eigenvalues ± 1. With the given choice of the additive constant, the Hamiltonian $\mathscr{H}$ has eigenvalue zero

$$\mathscr{H}\Phi_0 = 0, \tag{1c}$$

in the state Φ_0 in which all N spins are aligned along the negative z direction, that is,

$$\sigma_j^z \Phi_0 = -\Phi_0, \qquad j = 1, 2, \ldots, N. \tag{1d}$$

With the spin creation and annihilation operators $\sigma_j^\pm = \frac{1}{2}(\sigma_j^x \pm i\sigma_j^y)$ one has also

$$\sigma_j^- \Phi_0 = 0, \qquad j = 1, 2, \ldots, N. \tag{1e}$$

The eigenvectors Φ of $\mathscr{H}$ for which

$$\mathscr{H}\Phi = E\Phi \tag{2a}$$

can be chosen to correspond to a definite number n of spins reversed, so that

$$\Phi = \sum_{j_1 < \ldots < j_n} \varphi(j_1 \ldots j_n)\sigma_{j_1}^+ \ldots \sigma_{j_n}^+ \Phi_0. \tag{2b}$$

The eigenvalues E are of the form

$$E = \sum_{l=1}^{n} (1 - \cos K_l). \tag{2c}$$

Bethe's solution for the configuration space eigenfunctions $\varphi(j_1 \ldots j_n)$ which are defined in an ordered domain $j_1 < \ldots < j_n$, can be extended to symmetric functions

$$\varphi(\ldots j_l \ldots j_m \ldots) = \varphi(\ldots j_m \ldots j_l \ldots). \tag{2d}$$

These are invariant with respect to a change of coordinate from j_l to $j_l + N$ because of the periodic boundary conditions. Because of $(\sigma_j^+)^2 = 0$, which expresses the fact that there cannot be two $\frac{1}{2}$ spins reversed on the same lattice point, the most natural choice of the wavefunctions φ at coinciding arguments $j_l = j_m = j$ is

$$\varphi(\ldots j_l \ldots j_m \ldots)_{j_l = j_m = j} = 0. \tag{2e}$$

This condition will play some role in the following. In Bethe's approach for solving the equations in configuration space, extensions of the wavefunctions to the boundaries of the domain $j_1 < \ldots < j_n$ are also made use of.

The equations which result from (1*a*) and (2*a*) for the wavefunctions $\varphi(j_1 \ldots j_n)$ in configuration space, can be written in the form

$$(\boldsymbol{T} + \boldsymbol{V}^0 + \boldsymbol{V} - E)\varphi = 0 \tag{3a}$$

with a single-particle operator

$$\boldsymbol{T} = \sum_{l=1}^{m} T_l \tag{3b}$$

and two-particle operators

$$\boldsymbol{V}^0 = \sum_{(lm)} V_{lm}^0, \qquad \boldsymbol{V} = \sum_{(lm)} V_{lm} \tag{3c}$$

which are summed with respect to each pair (lm) of spins. These operators are given by the matrix elements

$$\langle j|T|j'\rangle = -\tfrac{1}{2}(\delta_{j+1,j'} + \delta_{j-1,j'} - 2\delta_{jj'}) \tag{3d}$$

$$\langle j_1 j_2|V^0|j_1' j_2'\rangle = \tfrac{1}{2}(\delta_{j_1 j_2} + \delta_{j_1' j_2'})\{(\delta_{j_1+1,j_1'} + \delta_{j_1-1,j_1'} - 2\delta_{j_1 j_1'})\delta_{j_2 j_2'} + \delta_{j_1 j_1'} \times (\delta_{j_2+1,j_2'} + \delta_{j_2-1,j_2'} - 2\delta_{j_2 j_2'})\} \tag{3e}$$

$$\langle j_1 j_2|V|j_1' j_2'\rangle = -\delta_{j_1 j_1'}\delta_{j_2 j_2'}(\delta_{j_1+1,j_2} + \delta_{j_1,j_2+1}). \tag{3f}$$

The operator V^0 has been chosen in such a way that the solutions of equation (3*a*) can satisfy the supplementary conditions (2*e*). The operator $T+V^0$ would correspond to the Hamiltonian of the XY model, whereas V is related to the nearest-neighbour interactions of the σ^z operators and would be multiplied by a parameter Δ in an asymmetric Hamiltonian. These configuration-space operators are symmetric and Hermitian, in contrast to operators frequently used in the literature, and can be immediately generalized to two and three dimensions.

The Fourier transforms of the matrix elements can be introduced by expanding the wavefunctions in terms of free single-particle spin waves according to

$$\varphi(j_1 \ldots j_n) = \sum_{k_1 \ldots k_n} \exp(\mathrm{i}k_1 j_1 + \ldots + \mathrm{i}k_n j_n)\phi_{k_1 \ldots k_n}, \tag{4a}$$

where the wavenumbers

$$k_l = \frac{2\pi}{N}\lambda_l, \qquad \lambda_l = 1, 2, \ldots, N \tag{4b}$$

are given by the conditions

$$\exp \mathrm{i}Nk_l = 1, \qquad l = 1, 2, \ldots, n. \tag{4c}$$

With the abbreviated notation

$$\epsilon_k = \cos k, \qquad \epsilon_{kk'} = \cos k + \cos k', \tag{5a}$$

the Fourier transforms of the expressions (3*d*, *e*, *f*) are given by

$$\langle k|T|k'\rangle = (1-\epsilon_k)\,\delta_{kk'} \tag{5b}$$

$$\langle k_1 k_2|V^0|k_1' k_2'\rangle = -\{(1-\epsilon_{k_1})+(1-\epsilon_{k_2})+(1-\epsilon_{k_1'})+(1-\epsilon_{k_2'})\}\frac{1}{N}\delta_{k_1+k_2,k_1'+k_2'} \tag{5c}$$

$$\langle k_1 k_2|V|k_1' k_2'\rangle = -\epsilon_{k_1-k_1',k_2-k_2'}\frac{1}{N}\delta_{k_1+k_2,k_1'+k_2'}. \tag{5d}$$

In writing the energy eigenvalue (2*c*) as

$$E = n - \epsilon_{K_1 \ldots K_n}, \tag{5e}$$

with

$$\epsilon_{K_1 \ldots K_n} = \cos K_1 + \ldots + \cos K_n \tag{5f}$$

and introducing the similar notation

$$\epsilon_{k_1 \ldots k_n} = \cos k_1 + \ldots + \cos k_n, \tag{5g}$$

the wave equation (3*a*) is transformed into

$$((T+V^0+V-E)\phi)_{k_1 \ldots k_n} = 0. \tag{6a}$$

with

$$((T-E)\phi)_{k_1 \ldots k_n} = (\epsilon_{K_1 \ldots K_n} - \epsilon_{k_1 \ldots k_n})\phi_{k_1 \ldots k_n} \tag{6b}$$

$$(V^0\phi)_{k_1 \ldots k_n} = \sum_{(lm)}\frac{1}{N}\sum_k \epsilon_{k,k_l+k_m-k}\phi_{k_1 \ldots k_l \to k \ldots k_m \to k_l+k_m-k \ldots k_n} \tag{6c}$$

$$(V\phi)_{k_1 \ldots k_n} = -\sum_{(lm)}\frac{1}{N}\sum_k \epsilon_{k_l-k,k_m-k}\phi_{k_1 \ldots k_l \to k \ldots k_m \to k_l+k_m-k \ldots k_n}. \tag{6d}$$

In deriving the expression (6c) of $V^0\phi$, the conditions

$$\frac{1}{N}\sum_k \phi_{k_1\ldots k_l\to k\ldots k_m\to k_m+k_l-k\ldots k_n} = 0 \tag{7}$$

have been used. These result from the conditions (2e) by Fourier transformation.

The contribution to $V\phi$ from a definite pair (lm) can be immediately obtained from that to $V^0\phi$ through the identity

$$\frac{\epsilon_{k_l-k,k_m-k}}{\epsilon_{k,k_l+k_m-k}} = \frac{\exp(\mathrm{i}k_l)+\exp(\mathrm{i}k_m)}{1+\exp\{\mathrm{i}(k_l+k_m)\}}. \tag{8a}$$

This is obtained by transforming the left-hand side according to

$$\frac{\cos(k_l-k)+\cos(k_m-k)}{\cos k+\cos(k_l+k_m-k)} = \frac{\cos\frac{1}{2}(k_l-k_m)}{\cos\frac{1}{2}(k_l+k_m)} \tag{8b}$$

which follows from $(k_l-k)+(k_m-k) = (-k)+(k_l+k_m-k)$, and shows that this ratio is independent of k. The right hand side of (8a) results after multiplying numerator and denominator by $\exp\{\frac{1}{2}\mathrm{i}(k_l+k_m)\}$.

Bethe's configuration-space wavefunction corresponds in wavenumber space to the symmetric wavefunction

$$\phi_{k_1\ldots k_n} = \sum_{P_K}\exp\left(-\tfrac{1}{2}\mathrm{i}\sum_{\bar{l}<\bar{m}}\theta_{\bar{l}\bar{m}}\right)\sum_{P_k}\chi_{k_1\ldots k_n} \tag{9a}$$

where

$$\chi_{k_1\ldots k_n} = \tilde{\chi}_{k_1\ldots k_n}\,\delta_{k_1+\ldots+k_n,K_1+\ldots+K_n} \tag{9b}$$

$$\tilde{\chi}_{k_1\ldots k_n} = \prod_{m'=1}^{n-1}\frac{1}{1-\exp(\mathrm{i}X_{m'})} \tag{9c}$$

with

$$X_1 = K_1-k_1, X_2 = K_1+K_2-k_1-k_2,\ldots, X_{n-1} = K_1+\ldots+K_{n-1}-k_1-\ldots-k_{n-1}. \tag{9d}$$

(The convention $X_0 = 0$ will also be used.) The summation over P_k means a summation over all $n!$ permutations of $k_1,\ldots,k_n$. It is convenient to separate the Kronecker delta in (9b) and refer occasionally to the factor $\tilde{\chi}_{k_1\ldots k_n}$, though this depends only on $n-1$ variables k. Both $\tilde{\chi}$ and χ are functions of the parameters $K_1,\ldots,K_n$, but this will not be explicitly indicated in general. The phases $\theta_{\bar{l}\bar{m}}$ are antisymmetric functions of $K_{\bar{l}}$, $K_{\bar{m}}$:

$$\theta_{\bar{l}\bar{m}} = \theta(K_{\bar{l}},K_{\bar{m}}) = -\theta_{\bar{m}\bar{l}} \tag{9e}$$

and the summation over P_K refers to a sum of all $n!$ permutations of $K_1,\ldots,K_n$.

The expression (9a–d) of the wavefunction can be obtained from the Bethe solution by Fourier transformation, though it represents only a main part resulting from such a transformation, and the vanishing of the remaining terms needs elaborate consideration. Here this expression will be considered as the ansatz for solving the equation (6a) and it will be shown directly that it satisfies the wave equation.

2. Conditions on the phase factors from the 'exclusion principle' in configuration space

A new feature of the solution revealed in k space is that the phase factors $\exp(-\frac{1}{2}i\theta)$ are determined from general considerations without reference to the detailed dynamics of the equations. The way in which this results can be unfolded more clearly in the simpler special cases $n = 2$ and $n = 3$; the case of general n is treated in the Appendix. There are two general relationships which determine the phase factors: the exclusion principle of spin $\frac{1}{2}$ reversals expressed by the conditions (2e) on the wavefunctions, and the special role of zero-momentum states in the case of the symmetric Hamiltonian (1a).

In the case $n = 2$, the conditions (2e) read

$$\varphi(j, j) = 0 \tag{10a}$$

and the Fourier transform (7) reduces to

$$\frac{1}{N}\sum_{k} \phi_{k,k_1+k_2-k} = 0. \tag{10b}$$

The wavefunction (9a) is of the form

$$\phi_{k_1k_2} = \sum_{P_K} \exp(-\tfrac{1}{2}i\theta_{12}) \sum_{P_k} \chi_{k_1k_2}, \tag{10c}$$

with

$$\sum_{P_k} \chi_{k_1k_2} = \left(\frac{1}{1-\exp\{i(K_1-k_1)\}} + \frac{1}{1-\exp\{i(K_1-k_2)\}}\right) \delta_{k_1+k_2,K_1+K_2}. \tag{10d}$$

The sum over P_K has two terms corresponding to the two permutations of K_1, K_2, with $\theta_{21} = -\theta_{12}$.

The summation in (10b) can be performed with the help of the identity

$$\frac{1}{N}\sum_{k} \frac{1}{1-\exp\{i(K_1-k)\}} = \frac{1}{1-\exp(iNK_1)}. \tag{11a}$$

This results by expressing

$$\frac{1}{1-\exp\{i(K_1-k)\}} = \frac{1}{1-\exp(iNK_1)} \sum_{j=0}^{N-1} \exp\{i(K_1-k)j\} \tag{11b}$$

with the help of a geometric series, summing term by term according to

$$\frac{1}{N}\sum_{k} \exp\{i(K-k)j\} = \delta_{j0} \tag{11c}$$

and writing $\exp(iNk) = 1$ in accordance with the condition (4c). With the help of this identity, equation (10b) with the expressions (10c, d) gives

$$\exp\left(-\frac{i}{2}\theta_{12}\right)\left(\frac{1}{1-\exp(iNK_1)} + \frac{1}{1-\exp(-iNK_2)}\right) + \exp\left(\frac{i}{2}\theta_{12}\right) \times \left(\frac{1}{1-\exp(iNK_2)} + \frac{1}{1-\exp(-iNK_1)}\right) = 0 \tag{12a}$$

for $k_1+k_2 = K_1+K_2$. Because of $\exp\{iN(k_1+k_2)\} = 1$, one has $\exp\{iN(K_1+K_2)\} = 1$

for $k_1+k_2 = K_1+K_2$, and the two terms are equal within both parentheses. This leads to

$$\exp(\mathrm{i}\theta_{12}) = -\frac{1-\exp(-\mathrm{i}NK_1)}{1-\exp(\mathrm{i}NK_1)} = \exp(-\mathrm{i}NK_1) \tag{12b}$$

or to

$$\exp \mathrm{i}(NK_1+\theta_{12}) = 1 \tag{12c}$$

$$\exp \mathrm{i}(NK_2+\theta_{21}) = 1. \tag{12d}$$

In the case $n = 3$, for the pair (12) the conditions (2*e*) read

$$\varphi(j, j, j_3) = 0, \tag{13a}$$

the Fourier transform of which gives

$$\frac{1}{N}\sum_k \phi_{k,k_1+k_2-k,k_3} = 0. \tag{13b}$$

The wavefunction is of the form

$$\phi_{k_1k_2k_3} = \sum_{P_K} \exp\{-\tfrac{1}{2}\mathrm{i}(\theta_{12}+\theta_{13}+\theta_{23})\} \sum_{P_k} \chi_{k_1k_2k_3}, \tag{14a}$$

with

$$\chi_{k_1k_2k_3} = \tilde{\chi}_{k_1k_2k_3}\,\delta_{k_1+k_2+k_3,K_1+K_2+K_3}, \tag{14b}$$

and

$$\tilde{\chi}_{k_1k_2k_3} = \frac{1}{1-\exp\{\mathrm{i}(K_1-k_1)\}}\,\frac{1}{1-\exp\{\mathrm{i}(K_1+K_2-k_1-k_2)\}} \tag{14c}$$

$$\tilde{\chi}_{k_3k_1k_2} = \frac{1}{1-\exp\{\mathrm{i}(K_1-k_3)\}}\,\frac{1}{1-\exp\{\mathrm{i}(K_1+K_2-k_3-k_1)\}} \tag{14d}$$

$$\tilde{\chi}_{k_2k_3k_1} = \frac{1}{1-\exp\{\mathrm{i}(K_1-k_2)\}}\,\frac{1}{1-\exp\{\mathrm{i}(K_1+K_2-k_2-k_3)\}}. \tag{14e}$$

It is sufficient to consider only these cyclic permutations of k_1, k_2, k_3 explicitly. The summation in (13*b*) can be performed immediately on the terms related to (14*c*) and (14*d*) which contain a summation variable only in one factor. In the term related to (14*e*) both factors would contain k, and it is convenient first to decompose $\tilde{\chi}_{k_2k_3k_1}$ as the sum of two terms:

$$\tilde{\chi}_{k_2k_3k_1} = \frac{1}{1-\exp\{\mathrm{i}(K_1+K_2-k_2-k_3)\}}\,\frac{1}{1-\exp\{-\mathrm{i}(K_2-k_3)\}} + \frac{1}{1-\exp\{\mathrm{i}(K_1-k_2)\}} \times \frac{1}{1-\exp\{\mathrm{i}(K_2-k_3)\}} \tag{14f}$$

so that the summation variable will appear only in one factor of each. The identity of (14*e*) with (14*f*) is more apparent as a special case of the identity (A.1*a*, *c*) of Appendix 1.

With the help of (11*a*) one obtains

$$\frac{1}{N}\sum_k \tilde{\chi}_{k,k_1+k_2-k,k_3} = \frac{1}{1-\exp(\mathrm{i}NK_1)}\,\frac{1}{1-\exp\{\mathrm{i}(K_1+K_2-k_1-k_2)\}} \tag{15a}$$

$$\frac{1}{N}\sum_k \tilde{\chi}_{k_3,k,k_1+k_2-k} = \frac{1}{1-\exp\{iN(K_1+K_2)\}}\,\frac{1}{1-\exp\{i(K_1-k_3)\}} \tag{15b}$$

$$\frac{1}{N}\sum_k \tilde{\chi}_{k_1+k_2-k,k_3,k} = \frac{1}{1-\exp\{iN(K_1+K_2)\}}\,\frac{1}{1-\exp\{-i(K_2-k_3)\}} + \frac{1}{1-\exp(iNK_1)} \times \frac{1}{1-\exp\{i(K_2-k_3)\}}. \tag{15c}$$

For $k_1+k_2+k_3 = K_1+K_2+K_3$, one can replace $K_1+K_2-k_1-k_2$ in (15*a*) by $-(K_3-k_3)$ and $N(K_1+K_2)$ in (15*b, c*) by $-NK_3$. In substituting (14*a*) into (13*b*), the contributions from (15*a, b, c*) remain unchanged if one performs a cyclic permutation $K_1 \to K_2$, $K_2 \to K_3$, $K_3 \to K_1$ in (15*b*) and in the first term of (15*c*), and in the related phase factors. After performing these operations, (15*b*) is changed into

$$\frac{1}{1-\exp(-iNK_1)}\,\frac{1}{1-\exp\{i(K_2-k_3)\}}, \tag{16a}$$

the first term of (15*c*) into

$$\frac{1}{1-\exp(-iNK_1)}\,\frac{1}{1-\exp\{-i(K_3-k_3)\}}, \tag{16b}$$

and the phase factor $\exp\{-\frac{1}{2}i(\theta_{12}+\theta_{13}+\theta_{23})\}$ into

$$\exp\{-\tfrac{1}{2}i(\theta_{23}+\theta_{21}+\theta_{31})\} = \exp\{-\tfrac{1}{2}i(\theta_{12}+\theta_{13}+\theta_{23})\}\exp\{i(\theta_{12}+\theta_{13})\}. \tag{16c}$$

In writing

$$\frac{1}{1-\exp(-iNK_1)} = -\frac{\exp(iNK_1)}{1-\exp(iNK_1)} \tag{16d}$$

in (16*a, b*), the contribution of the terms with cyclic permutations P_k(cycl) of $k_1k_2k_3$ to $(1/N)\sum_k \phi_{k,k_1+k_2-k,k_3}$ results in the form

$$\sum_{P_K}\exp\{-\tfrac{1}{2}i(\theta_{12}+\theta_{13}+\theta_{23})\}\frac{1}{N}\sum_k\sum_{P_k(\text{cycl})}\chi_{k,k_1+k_2-k,k_3}$$
$$= \sum_{P_K}\frac{\exp\{-\frac{1}{2}i(\theta_{12}+\theta_{13}+\theta_{23})\}}{1-\exp(iNK_1)}\left(\frac{1}{1-\exp\{-i(K_3-k_3)\}}+\frac{1}{1-\exp\{i(K_2-k_3)\}}\right) \times[1-\exp\{i(NK_1+\theta_{12}+\theta_{13})\}]. \tag{16e}$$

This expression vanishes identically if one has

$$\exp i(NK_1+\theta_{12}+\theta_{13}) = 1 \tag{17a}$$

$$\exp i(NK_2+\theta_{23}+\theta_{21}) = 1 \tag{17b}$$

$$\exp i(NK_3+\theta_{31}+\theta_{32}) = 1. \tag{17c}$$

Conversely, if (16*e*) is to vanish for all values of k_3, and $K_1 \neq K_2 \neq K_3$, the equations (17*a, b, c*) follow. If one replaces the summation over cyclic permutations of k_1, k_2, k_3 on the left hand side of (16*e*) by all permutations, the equations (17*a, b, c*) and the vanishing of (16*e*) follows similarly from equations (13*b*) which are to be satisfied for all k_1, k_2, k_3. The derivation shows that (16*e*) vanishes also if one restricts the summation over all permutations of K_1, K_2, K_3 to cyclic permutations.

In the general case of the wavefunction (9*a*), the conditions (7) lead to the identities

$$\exp \mathrm{i}\left(NK_{\bar{l}}+\sum_{\bar{m}=1}^{n}\theta_{\bar{l}\bar{m}}\right)=1, \qquad \text{for } \bar{l}=1,2,\ldots,n. \tag{18}$$

This is shown in Appendix 2. With the convention $\theta_{\bar{l}\bar{l}}=0$ the absence of the term with $\bar{m}=\bar{l}$ need not be indicated explicitly.

In Bethe's derivation, these conditions result by imposing the periodicity conditions on the configuration-space wavefunctions given in the domain $j_1<\ldots<j_n$. The transforms (4*a*) of $\phi_{k_1\ldots k_n}$ are periodic because of the conditions (4*b*, *c*) on the wavevectors k_l, so that the periodicity conditions are by definition satisfied for any wavefunction $\phi_{k_1\ldots k_n}$ described in wavevector space. The related conditions follow here from the 'exclusion principle' represented by the equations (2*e*) and (7).

2. Conditions on the phase factors from the zero-momentum degeneracy

Because of its rotational symmetry, the Hamiltonian (1*a*) commutes with the symmetrized spin-raising operator

$$S^+=\sum_{j=1}^{N}\sigma_j^+, \tag{19a}$$

so that one has

$$[\mathscr{H},S^+]=0. \tag{19b}$$

Accordingly, from the equation $\mathscr{H}\Phi=E\Phi$ follows

$$\mathscr{H}(S^+\Phi)=E(S^+\Phi). \tag{19c}$$

This shows that if Φ is an eigenvector of $\mathscr{H}$, then $S^+\Phi$, and similarly $(S^+)^m\Phi$, are eigenvectors of $\mathscr{H}$ with the same eigenvalue E.

A single-spin wave solution

$$\varphi_K(j)=\exp(\mathrm{i}Kj), \qquad \exp(\mathrm{i}KN)=1, \tag{20a}$$

of the wave equation in configuration space defines in this way a symmetric two-spin wave solution

$$\varphi_0(j_1j_2)=\varphi_K(j_1)+\varphi_K(j_2). \tag{20b}$$

This has the same energy $E=1-\cos K$ as $\varphi_K(j)$ and differs from it by the addition of a spin wave of wavevector zero, represented by the creation operator (19*a*).

Any two-spin wave eigenfunction $\varphi(j_1j_2)$ of the Hamiltonian which corresponds to a different energy eigenvalue is orthogonal to $\varphi_0(j_1j_2)$, so that one has

$$\sum_{j_1j_2}\varphi_0^*(j_1j_2)\varphi(j_1j_2)=0. \tag{20c}$$

Because of the symmetry $\varphi(j_2j_1)=\varphi(j_1j_2)$ of the wavefunction this can also be written in the form

$$\sum_{j_2}\varphi_K^*(j_2)\sum_{j_1}\varphi(j_1j_2)=0. \tag{20d}$$

If the energy of the state $\varphi(j_1j_2)$ is different from all the single particle energies

$E = 1 - \cos K$, then according to this equation there is a complete set $\varphi_K(j)$ of single-particle states (20a) to which the single-particle wavefunction $\Sigma_{j_1} \varphi(j_1 j_2)$ is orthogonal. Consequently, this function must vanish and one has

$$\sum_{j_1} \varphi(j_1 j_2) = 0. \tag{20e}$$

In terms of the Fourier transform $\phi_{k_1 k_2}$ this means

$$\phi_{k_1 = 0, k_2} = 0. \tag{20f}$$

Those two-particle eigenstates which are nondegenerate with the single-particle solutions have no zero-momentum component.

A similar argument holds for a general n-spin wave eigenfunction. Denoting by $\varphi(j_2 \ldots j_n)$ an energy eigenfunction of the $(n-1)$-spin problem, the symmetrized n-spin wavefunction

$$\varphi_0(j_1 \ldots j_n) = \varphi_K(j_2 \ldots j_n) + \varphi_K(j_1 j_3 \ldots j_n) + \ldots + \varphi_K(j_1 \ldots j_{n-1}) \tag{21a}$$

is an eigenfunction of the n-spin problem with the same energy eigenvalue. An n-spin wave solution which belongs to a different energy is orthogonal to $\varphi_0(j_1 \ldots j_n)$, and one has

$$\sum_{j_1 \ldots j_n} \varphi_0^*(j_1 \ldots j_n) \varphi(j_1 \ldots j_n) = 0. \tag{21b}$$

From this one concludes as before

$$\sum_{j_2 \ldots j_n} \varphi_K^*(j_2 \ldots j_n) \sum_{j_1} \varphi(j_1 \ldots j_n) = 0. \tag{21c}$$

If the energy of the state $\varphi(j_1 \ldots j_n)$ is different from all the energy eigenvalues of the $(n-1)$-spin wave states, then there is a complete set of $(n-1)$-spin eigenfunctions $\varphi_K(j_2 \ldots j_n)$ to which $\Sigma_{j_1} \varphi(j_1 \ldots j_n)$ is orthogonal. One has consequently

$$\sum_{j_1} \varphi(j_1 \ldots j_n) = 0, \tag{21d}$$

and for the Fourier transform $\phi_{k_1 \ldots k_n}$ one obtains

$$\phi_{k_1 = 0, k_2 \ldots k_n} = 0. \tag{21e}$$

This zero-momentum condition, together with the equations (18), gives all the necessary restrictions on the phase factors in the ansatz (9a) of the energy eigenfunctions.

In the case $n = 2$, for $k_1 = 0$ one can write $k_2 = K_1 + K_2$ because of

$$k_1 + k_2 = K_1 + K_2,$$

so that the condition to be satisfied is

$$\phi_{0, K_1 + K_2} = 0. \tag{22a}$$

With the form (10c, d) of the two-spin wave eigenfunctions, this gives

$$\exp\left(-\frac{i}{2}\theta_{12}\right)\left(\frac{1}{1-\exp(iK_1)} + \frac{1}{1-\exp(-iK_2)}\right) + \exp\left(\frac{i}{2}\theta_{12}\right) \times \left(\frac{1}{1-\exp(iK_2)} + \frac{1}{1-\exp(-iK_1)}\right) = 0, \tag{22b}$$

or

$$\exp(-i\theta_{12}) = -\frac{1+\exp\{-i(K_1+K_2)\}-2\exp(-iK_2)}{1+\exp\{-i(K_1+K_2)\}-2\exp(-iK_1)}. \tag{22c}$$

In the case $n = 3$, one can consider the expressions (14*c*, *e*) of $\tilde{\chi}_{k_1k_2k_3}$ and $\tilde{\chi}_{k_2k_3k_1}$, together with the expression of $\tilde{\chi}_{k_2k_1k_3}$. With the help of the identity (A.1*a*, *c*) of Appendix 1 the latter can be decomposed in the form

$$\tilde{\chi}_{k_2k_1k_3} = \frac{1}{1-\exp\{i(K_1-k_2)\}}\,\frac{1}{1-\exp\{i(K_2-k_1)\}}+\frac{1}{1-\exp\{-i(K_2-k_1)\}} \times\frac{1}{1-\exp\{i(K_1+K_2-k_1-k_2)\}}. \tag{23a}$$

If one replaces $K_1+K_2-k_2-k_3$ by $-(K_3-k_1)$ in (14*e*), one obtains, for $k_1+k_2+k_3 = K_1+K_2+K_3$

$$\begin{aligned}&\tilde{\chi}_{k_1k_2k_3}+\tilde{\chi}_{k_2k_1k_3}+\tilde{\chi}_{k_2k_3k_1}\\ &= \left(\frac{1}{1-\exp\{i(K_1-k_1)\}}+\frac{1}{1-\exp\{-i(K_2-k_1)\}}\right)\frac{1}{1-\exp\{i(K_1+K_2-k_1-k_2)\}}\\ &\quad+\left(\frac{1}{1-\exp\{i(K_2-k_1)\}}+\frac{1}{1-\exp\{-i(K_3-k_1)\}}\right)\frac{1}{1-\exp\{i(K_1-k_3)\}}.\end{aligned} \tag{23b}$$

The sum $\Sigma_{P_k}\tilde{\chi}_{k_1k_2k_3}$ over the six permutations P_k of k_1, k_2, k_3 results from (23*b*) by adding to it the expression obtained by an interchange of k_2 and k_3.

If in the expression (9*a*–*d*) of $\phi_{k_1\ldots k_n}$ one replaces the sum over all permutations P_K of K_1, K_2, K_3 by cyclic permutations P_K(cycl) and transpositions which are explicitly indicated, and puts $k_1 = 0$, the equation $\phi_{k_1=0,k_2k_3} = 0$ can be rewritten with the help of (23*b*) in the form

$$\begin{aligned}&\sum_{P_K(\mathrm{cycl})}\exp\left(-\frac{i}{2}(\theta_{13}+\theta_{23})\right)\left\{\exp\left(-\frac{i}{2}\theta_{12}\right)\left(\frac{1}{1-\exp(iK_1)}+\frac{1}{1-\exp(-iK_3)}\right)\right.\\ &\left.+\exp\left(\frac{i}{2}\theta_{12}\right)\left(\frac{1}{1-\exp(iK_2)}+\frac{1}{1-\exp(-iK_1)}\right)\right\}\left(\frac{1}{1-\exp\{i(K_1+K_2-k_2)\}}+(k_2\leftrightarrow k_3)\right)\\ &+\sum_{P_K(\mathrm{cycl})}\exp\left(-\frac{i}{2}(\theta_{12}+\theta_{13})\right)\left\{\exp\left(-\frac{i}{2}\theta_{23}\right)\left(\frac{1}{1-\exp(iK_2)}+\frac{1}{1-\exp(-iK_3)}\right)\right.\\ &\left.+\exp\left(\frac{i}{2}\theta_{23}\right)\left(\frac{1}{1-\exp(iK_3)}+\frac{1}{1-\exp(-iK_2)}\right)\right\}\left(\frac{1}{1-\exp\{i(K_1-k_2)\}}+(k_2\leftrightarrow k_3)\right) = 0.\end{aligned} \tag{23c}$$

If equations (22*b*) held for each pair of K_l, K_m, or

$$\exp(-i\theta_{lm}) = -\frac{1+\exp\{-i(K_l+K_m)\}-2\exp(-iK_m)}{1+\exp\{-i(K_l+K_m)\}-2\exp(-iK_l)} \tag{23d}$$

for $l \neq m = 1, 2, 3$, the equation (23*c*) is evidently satisfied, since all the corresponding brackets vanish separately. Conversely, since the equations (23*c*) are to be satisfied for all values of k_2, k_3, these brackets must vanish, and the relationships (23*d*) follow.

As shown in Appendix 3, the same relationships follow from the equation

$$\phi_{k_1=0,k_2\ldots k_n} = 0$$

in the general case, for $l \neq m = 1, 2, \ldots, n$. In Bethe's derivation in configuration space, the validity of these equations is concluded from a consideration of nearest-neighbour terms together with an extrapolation of the wavefunctions to the borders of the domain $j_1 < \ldots < j_n$. In the present derivation, this expression of the phase factors results from simple symmetry requirements.

4. The two-spin wave solutions

In the case $n = 2$, the equations (6*a*, *b*, *c*, *d*) simplify to

$$((T + V^0 + V - E)\phi)_{k_1k_2} = 0, \tag{24a}$$

with

$$((T - E)\phi)_{k_1k_2} = (\epsilon_{K_1K_2} - \epsilon_{k_1k_2})\phi_{k_1k_2} \tag{24b}$$

$$(V^0\phi)_{k_1k_2} = \frac{1}{N}\sum_k \epsilon_{k,k_1+k_2-k}\phi_{k,k_1+k_2-k} \tag{24c}$$

$$(V\phi)_{k_1k_2} = -\frac{1}{N}\sum_k \epsilon_{k_1-k,k_2-k}\phi_{k,k_1+k_2-k}. \tag{24d}$$

In order to show that the function (10*c*, *d*) gives a solution of the wave equation (24*a*), some identities can be referred to. The identity

$$(\cos k - \cos K_1)\frac{1}{1-\exp\{i(K_1-k)\}} = -\tfrac{1}{2}\{\exp(-iK_1) - \exp(ik)\} \tag{25a}$$

can be immediately obtained by multiplying with the denominator of the left hand side. If one replaces in it K_1 by $K_1 - k_1 - k_2$ and k by $k - k_1 - k_2$, one can write similarly

$$\{\cos(k_1+k_2-k) - \cos(k_1+k_2-K_1)\}\frac{1}{1-\exp\{i(K_1-k)\}}$$
$$= -\tfrac{1}{2}[\exp\{-i(K_1-k_1-k_2)\} - \exp\{i(k-k_1-k_2)\}]. \tag{25b}$$

If one adds the two equations and sums with respect to k, with the help of the identity

$$\frac{1}{N}\sum_k \exp(ik) = 0, \tag{25c}$$

one obtains

$$\frac{1}{N}\sum_k (\epsilon_{k,k_1+k_2-k} - \epsilon_{K_1,k_1+k_2-K_1})\frac{1}{1-\exp\{i(K_1-k)\}} = -\tfrac{1}{2}\exp(-iK_1)[1+\exp\{i(k_1+k_2)\}]. \tag{25d}$$

With the notation

$$\tilde{\chi}_{k_1k_2} = \frac{1}{1-\exp\{i(K_1-k_1)\}} \tag{26a}$$

this gives immediately

$$V^0\tilde{\chi}_{k_1k_2} = -\tfrac{1}{2}\exp(-\mathrm{i}K_1)[1+\exp\{\mathrm{i}(k_1+k_2)\}]+\dots \tag{26b}$$

where the dots indicate a term proportional to $\epsilon_{K_1,k_1+k_2-K_1}$ which does not contribute to (24*c*), because of the relationship (10*b*). Through the identity (8*a*), one has also

$$V\tilde{\chi}_{k_1k_2} = \tfrac{1}{2}\exp(-\mathrm{i}K_1)\{\exp(\mathrm{i}k_1)+\exp(\mathrm{i}k_2)\}+\dots \tag{26c}$$

with a similar interpretation of the dots.

In order to calculate (24*b*), note the identity

$$\begin{aligned}\epsilon_{K_1K_2}-\epsilon_{k_1k_2} &= \cos K_1+\cos K_2-\cos k_1-\cos k_2\\ &= \tfrac{1}{2}[\![\exp(-\mathrm{i}K_1)[1+\exp\{\mathrm{i}(k_1+k_2)\}]-[1+\exp\{-\mathrm{i}(K_1+K_2)\}]\exp(\mathrm{i}k_1)]\!]\\ &\quad\times[1-\exp\{\mathrm{i}(K_1-k_1)\}]\end{aligned} \tag{27a}$$

valid for $k_1+k_2 = K_1+K_2$, which can be checked by performing the multiplications. If one multiplies this by $\tilde{\chi}_{k_1k_2}$, the factor $[1-\exp\{\mathrm{i}(K_1-k_1)\}]$ cancels, and one obtains

$$\begin{aligned}(\epsilon_{K_1K_2}-\epsilon_{k_1k_2})\tilde{\chi}_{k_1k_2} &= \tfrac{1}{2}\exp(-\mathrm{i}K_1)[1+\exp\{\mathrm{i}(k_1+k_2)\}]\\ &\quad-\tfrac{1}{2}[1+\exp\{-\mathrm{i}(K_1+K_2)\}]\exp(\mathrm{i}k_1).\end{aligned} \tag{27b}$$

The first term of (27*b*) can be seen to cancel immediately the main contribution of the expression (26*b*) of $V^0\tilde{\chi}_{k_1k_2}$. If one interchanges the role of k_1 and k_2, the second term of (27*b*) will contain a factor $\exp(\mathrm{i}k_2)$ instead of $\exp(\mathrm{i}k_1)$, whereas the right hand side of (26*c*) remains unchanged. One obtains accordingly

$$\begin{aligned}(\boldsymbol{T}+\boldsymbol{V}^0+\boldsymbol{V}-E)(\tilde{\chi}_{k_1k_2}+\tilde{\chi}_{k_2k_1}) &= -\tfrac{1}{2}[1+\exp\{-\mathrm{i}(K_1+K_2)\}-2\exp(-\mathrm{i}K_1)]\\ &\quad\times\{\exp(\mathrm{i}k_1)+\exp(\mathrm{i}k_2)\}+\dots.\end{aligned} \tag{27c}$$

With the form (10*c*, *d*) of $\phi_{k_1k_2}$ one concludes that the wave equation (24*a*) is satisfied if

$$\begin{aligned}&\exp(-\tfrac{1}{2}\mathrm{i}\theta_{12})[1+\exp\{-\mathrm{i}(K_1+K_2)\}-2\exp(-\mathrm{i}K_1)]+\exp(\tfrac{1}{2}\mathrm{i}\theta_{12})\\ &\quad\times[1+\exp\{-\mathrm{i}(K_1+K_2)\}-2\exp(-\mathrm{i}K_2)] = 0.\end{aligned} \tag{28}$$

This is, however, only another form of the relationship (22*c*) which follows from the condition (22*a*).

5. The three-spin wave solutions

For the three-spin wave case, the equations corresponding to (6*a*, *b*, *c*, *d*) are

$$((\boldsymbol{T}+\boldsymbol{V}^0+\boldsymbol{V}-E)\phi)_{k_1k_2k_3} = 0 \tag{29a}$$

$$((\boldsymbol{T}-E)\phi)_{k_1k_2k_3} = (\epsilon_{K_1K_2K_3}-\epsilon_{k_1k_2k_3})\phi_{k_1k_2k_3} \tag{29b}$$

$$\begin{aligned}(\boldsymbol{V}^0\phi)_{k_1k_2k_3} = \frac{1}{N}\sum_k (&\epsilon_{k,k_1+k_2-k}\phi_{k,k_1+k_2-k,k_3}+\epsilon_{k,k_1+k_3-k}\phi_{k,k_2,k_1+k_3-k}\\ &+\epsilon_{k,k_2+k_3-k}\phi_{k_1,k,k_2+k_3-k})\end{aligned} \tag{29c}$$

$$(V\phi)_{k_1k_2k_3} = -\frac{1}{N}\sum_k (\epsilon_{k_1-k,k_2-k}\phi_{k,k_1+k_2-k,k_3} + \epsilon_{k_1-k,k_3-k}\phi_{k,k_2,k_1+k_3-k}$$
$$+\epsilon_{k_2-k,k_3-k}\phi_{k_1,k,k_2+k_3-k}). \tag{29d}$$

One has to verify that the wavefunction (14*a*, *b*, *c*) satisfies the wave equation (29*a*). This case is still sufficiently simple to be written out in detail and will illustrate some of the features of the method used in the general case.

In order to calculate the terms of $V^0\phi$, one can apply the relationship (25*d*). With the wavefunction (14*c*) one obtains

$$\frac{1}{N}\sum_k \epsilon_{k,k_1+k_2-k}\tilde{\chi}_{k,k_1+k_2-k,k_3} = -\tfrac{1}{2}\exp(-\mathrm{i}K_1)[1+\exp\{\mathrm{i}(k_1+k_2)\}]$$
$$\times\frac{1}{1-\exp\{\mathrm{i}(K_1+K_2-k_1-k_2)\}}+\dots \tag{30a}$$

$$\frac{1}{N}\sum_k \epsilon_{k,k_2+k_3-k}\tilde{\chi}_{k_1,k,k_2+k_3-k} = -\tfrac{1}{2}\exp(-\mathrm{i}K_2)[1+\exp\{\mathrm{i}(k_2+k_3)\}]\exp\{-\mathrm{i}(K_1-k_1)\}$$
$$\times\frac{1}{1-\exp\{\mathrm{i}(K_1-k_1)\}}+\dots. \tag{30b}$$

As before the dots in (30*a*) indicate terms proportional to $\epsilon_{K_1,k_1+k_2-K_1}$ and the sum of all similar contributions will be shown to vanish. In calculating

$$\frac{1}{N}\sum_k \epsilon_{k,k_1+k_3-k}\tilde{\chi}_{k,k_2,k_1+k_3-k} \tag{30c}$$

both factors of (14*c*) will depend on k, and a decomposition similar to (14*f*) is first to be performed. With

$$\tilde{\chi}_{k,k_2,k_1+k_3-k} = \frac{1}{1-\exp\{\mathrm{i}(K_1-k)\}}\,\frac{1}{1-\exp\{\mathrm{i}(K_2-k_2)\}}$$
$$+\frac{1}{1-\exp\{\mathrm{i}(K_1+K_2-k-k_2)\}}\,\frac{1}{1-\exp\{-\mathrm{i}(K_2-k_2)\}} \tag{30d}$$

one obtains for (30*c*)

$$-\tfrac{1}{2}\exp(-\mathrm{i}K_1)[1+\exp\{\mathrm{i}(k_1+k_3)\}]\left(\frac{1}{1-\exp\{\mathrm{i}(K_2-k_2)\}}+\frac{\exp\{-\mathrm{i}(K_2-k_2)\}}{1-\exp\{-\mathrm{i}(K_2-k_2)\}}\right)+\dots. \tag{30e}$$

Since the expression in the large parentheses vanishes identically, the function $\tilde{\chi}_{k_1k_2k_3}$ will give no contribution to the term with ϵ_{k,k_1+k_3-k} of $V^0\phi$. It will contribute only to the interaction between its 'adjacent pairs' (12) and (23).

The vanishing of the sum of contributions from the terms indicated by dots and of all similar contributions can be seen by considering the contributions to one term in (29*c*), say to that of the pair (12), with factor ϵ_{k,k_1+k_2-k}. For cyclic permutations of k_1, k_2, k_3 in $\tilde{\chi}$, the terms indicated by dots are proportional to the terms on the right hand sides of equations (15*a*, *b*, *c*). The right hand side of (15*a*) and the second term of (15*c*) appear multiplied by a factor $-\epsilon_{K_1,k_1+k_2-K_1}$, whereas (15*b*) and the first term of (15*c*) obtain a factor $-\epsilon_{K_1+K_2-k_3,k_1+k_2+k_3-K_1-K_2}$. Taking into account the Kronecker delta of $\chi_{k_1k_2k_3}$ and performing the permutations of K_1, K_2, K_3 which lead to the changes

(16*a*, *b*, *c*), this last factor also becomes $-\epsilon_{K_1,k_1+k_2-K_1}$ and the whole expression (16*e*) appears multiplied by this factor. This vanishes if the equations (17*a*, *b*, *c*) are satisfied, that is because of the identity (13*b*). The same equations (17*a*, *b*, *c*) would follow also from the requirement that the wavefunction (14*a*, *b*, *c*) should satisfy equation (29*a*).

The contributions to the terms of $V\phi$ can be obtained according to the relationship (8*a*), by replacing the factor $[1+\exp\{i(k_1+k_2)\}]$ in (30*a*) by $\{\exp(ik_1)+\exp(ik_2)\}$, and by a similar replacement in (30*b*).

In calculating (29*b*), terms of the form

$$((T-E)\tilde{\chi})_{k_1k_2k_3} = (\epsilon_{K_1K_2K_3}-\epsilon_{k_1k_2k_3})\tilde{\chi}_{k_1k_2k_3} \tag{31a}$$

are to be considered. For $k_1+k_2+k_3 = K_1+K_2+K_3$, one can write

$$\epsilon_{K_1K_2K_3}-\epsilon_{k_1k_2k_3} = (\epsilon_{K_1,k_1+k_2-K_1}-\epsilon_{k_1k_2})+(\epsilon_{K_2K_3}-\epsilon_{k_3,K_2+K_3-k_3}). \tag{31b}$$

On the right hand side of this, the first term can be written, according to the identity (27*a*), in a form containing a factor $[1-\exp\{i(K_1-k_1)\}]$, and the second term in a form with a factor $[1-\exp\{i(K_1+K_2-k_1-k_2)\}]$. These will cancel the corresponding factors of $\tilde{\chi}_{k_1k_2k_3}$, so that for (31*a*) one can write

$$\begin{aligned}(\epsilon_{K_1K_2K_3}&-\epsilon_{k_1k_2k_3})\tilde{\chi}_{k_1k_2k_3}\\ &= \tfrac{1}{2}[\![\exp(-iK_1)[1+\exp\{i(k_1+k_2)\}]-[1+\exp\{-i(K_1+K_2)\}]\exp(ik_1)]\!]\\ &\qquad\times\frac{1}{1-\exp\{i(K_1+K_2-k_1-k_2)\}}\\ &\qquad+\tfrac{1}{2}[\![\exp(-iK_2)[1+\exp\{i(k_2+k_3)\}]-[1+\exp\{-i(K_2+K_3)\}]\exp(ik_2)]\!]\\ &\qquad\times\exp\{-i(K_1-k_1)\}\frac{1}{1-\exp\{i(K_1-k_1)\}}.\end{aligned} \tag{31c}$$

The first part of both terms can be seen to cancel exactly the main contributions to $V^0\tilde{\chi}$ from the expressions (30*a*, *b*). In the second parts, $\exp(ik_1)$ can be replaced by

$$\tfrac{1}{2}\{\exp(ik_1)+\exp(ik_2)\}$$

and $\exp(ik_2)$ by $\tfrac{1}{2}\{\exp(ik_2)+\exp(ik_3)\}$, in the symmetrized sum of $\tilde{\chi}_{k_1k_2k_3}$ over permutations of k_1, k_2, k_3. Adding all the contributions, one obtains

$$\begin{aligned}(T+V^0+V-E)&\sum_{P_k}\chi_{k_1k_2k_3}\\ = -\tfrac{1}{4}\sum_{P_k}&\Bigg([1+\exp\{-i(K_1+K_2)\}-2\exp(-iK_1)]\{\exp(ik_1)+\exp(ik_2)\}\\ &\times\frac{1}{1-\exp\{i(K_1+K_2-k_1-k_2)\}}+[1+\exp\{-i(K_2+K_3)\}-2\exp(-iK_2)]\\ &\times\{\exp(ik_2)+\exp(ik_3)\}\exp\{-i(K_1-k_1)\}\frac{1}{1-\exp\{i(K_1-k_1)\}}\Bigg)+\dots\end{aligned} \tag{32}$$

The expression (14*a*) of $\phi_{k_1k_2k_3}$ can be seen to be a solution of the wave equation (29*a*), if equations of the type (28) are satisfied for all pairs K_l, K_m, $l \neq m = 1, 2, 3$, that is, if the phase factors satisfy equations (23*d*).

6. The general solution

In the interaction terms (6*c*, *d*) of the wave equation (6*a*) for general n, there is no restriction on choosing a pair of indices (lm) and one can have $l \lessgtr m$. Assume now, however, for definiteness that

$$1 \leqslant l < m \leqslant n.$$

(When $l > m$, the treatment is similar.) The functions entering into the definition (9*a*–*d*) of the solutions $\phi_{k_1 \ldots k_n}$ of the wave equation can then be factorized in the form

$$\tilde{\chi}_{k_1 \ldots k_n} = \left(\prod_{m'=1}^{l-1} \frac{1}{1-\exp(\mathrm{i}X_{m'})}\right)\left(\prod_{m'=l}^{m-1} \frac{1}{1-\exp(\mathrm{i}X_{m'})}\right)\left(\prod_{m'=m}^{n-1} \frac{1}{1-\exp(\mathrm{i}X_{m'})}\right) \tag{33a}$$

where the first product contains neither k_l nor k_m, the second product contains factors with k_l only, and the third product contains k_l and k_m only through the sum k_l+k_m. In the interaction terms (6*c*, *d*) the summation over k will affect only the second product.

In investigating the interaction terms, two cases are to be considered. In case (1) the indices l and m are adjacent and $m = l+1$, in case (2) they are separated and $m > l+1$.

In case (1) there is only one factor in the second product, namely

$$\frac{1}{1-\exp(\mathrm{i}X_l)}. \tag{33b}$$

Since $X_l = X_{l-1}+K_l-k_l$, the corresponding term in the expression (6*c*) of $V^0\phi$ contains a contribution with a factor

$$\frac{1}{N}\sum_k \epsilon_{k,k_l+k_{l+1}-k}\frac{1}{1-\exp(\mathrm{i}X_{l-1}+K_l-k)}$$
$$= -\tfrac{1}{2}\exp(-\mathrm{i}K_l)[1+\exp\{\mathrm{i}(k_l+k_{l+1})\}]\exp(-\mathrm{i}X_{l-1})+\ldots. \tag{33c}$$

The summation is performed in the same way as in expression (26*b*) for $n = 2$, and the dots again indicate terms the sum of which will give a vanishing contribution.

In case (2), the second product in (33*a*) contains more than one factor with k_l. According to the identity (A.1*a*, *b*) proved in Appendix 1, it can be decomposed, however, in the form

$$\prod_{m'=l}^{m-1}\frac{1}{1-\exp(\mathrm{i}X_{m'})} = \sum_{m'=l}^{m-1}\frac{1}{1-\exp(\mathrm{i}X_{m'})}\prod_{l' \neq m'}^{m-1}\frac{1}{1-\exp\{\mathrm{i}(X_{l'}-X_{m'})\}} \tag{33d}$$

in which each term depends on k_l only through its first factor. After replacing k_l by k, and accordingly $X_{m'}$ by $X_{m'}+k_l-k$, multiplying by an energy factor and performing the summation, the corresponding contribution to $V^0\phi$ will contain a factor

$$\frac{1}{N}\sum_k \epsilon_{k,k_l+k_m-k}\prod_{m'=l}^{m-1}\frac{1}{1-\exp\{\mathrm{i}(X_{m'}+k_l-k)\}}$$
$$= -\tfrac{1}{2}\exp(-\mathrm{i}k_l)[1+\exp\{\mathrm{i}(k_l+k_m)\}]\sum_{m'=l}^{m-1}\exp(-\mathrm{i}X_{m'})\prod_{l'\neq m'}^{m-1}\frac{1}{1-\exp\{\mathrm{i}(X_{l'}-X_{m'})\}}+\ldots. \tag{33e}$$

According to the identity (A.2*a*) of Appendix 1, however, the first term vanishes identically, and the only remaining contribution of (33*a*) is through the terms indicated by dots.

It is shown in Appendix 2 that the sum of all contributions indicated by dots both to $V^0\phi$ and to $V\phi$ is zero. The only non-vanishing contribution to the interactions from a term $\chi_{k_1\ldots k_n}$ in the wavefunction comes accordingly from adjacent pairs $(l, l+1)$ of case (1).

From the definition (9*b*, *c*) of $\chi_{k_1\ldots k_n}$, the function defined by

$$\chi^{(l)}_{k_1\ldots k_n} = \{1-\exp(\mathrm{i}X_l)\}\chi_{k_1\ldots k_n} \tag{34a}$$

differs from χ only through the omission of the lth factor of $\tilde{\chi}$. Accordingly, we can write

$$(V^0\chi)_{k_1\ldots k_n} = -\tfrac{1}{2}\sum_{l=1}^{n-1} \exp(-\mathrm{i}K_l)[1+\exp\{\mathrm{i}(k_l+k_{l+1})\}]\exp(-\mathrm{i}X_{l-1})\chi^{(l)}_{k_1\ldots k_n}+\ldots. \tag{34b}$$

From the relationship (8*a*), one obtains similarly

$$(V\chi)_{k_1\ldots k_n} = \tfrac{1}{2}\sum_{l=1}^{n-1} \exp(-\mathrm{i}K_l)\{\exp(\mathrm{i}k_l)+\exp(\mathrm{i}k_{l+1})\}\exp(-\mathrm{i}X_{l-1})\chi^{(l)}_{k_1\ldots k_n}+\ldots. \tag{34c}$$

The expression

$$((T-E)\chi)_{k_1\ldots k_n} = (\epsilon_{K_1\ldots K_n}-\epsilon_{k_1\ldots k_n})\chi_{k_1\ldots k_n} \tag{34d}$$

can be transformed with the help of the identity

$$\epsilon_{K_1\ldots K_n}-\epsilon_{k_1\ldots k_n} = \tfrac{1}{2}\sum_{l=1}^{n-1} [\![\exp(-\mathrm{i}K_l)[1+\exp\{\mathrm{i}(k_l+k_{l+1})\}]-[1+\exp\{-\mathrm{i}(K_l+K_{l+1})\}] \times\exp(\mathrm{i}k_l)]\!]\exp(-\mathrm{i}X_{l-1})\{1-\exp(\mathrm{i}X_l)\} \tag{35a}$$

which is valid for $k_1+\ldots+k_n = K_1+\ldots+K_n$ and is proved in Appendix 4. One obtains

$$(\epsilon_{K_1\ldots K_n}-\epsilon_{k_1\ldots k_n})\chi_{k_1\ldots k_n} = \tfrac{1}{2}\sum_{l=1}^{n-1} [\![\exp(-\mathrm{i}K_l)[1+\exp\{\mathrm{i}(k_l+k_{l+1})\}] -[1+\exp\{-\mathrm{i}(K_l+K_{l+1})\}]\exp(\mathrm{i}k_l)]\!]\exp(-iX_{l-1})\chi^{(l)}_{k_1\ldots k_n}. \tag{35b}$$

The first part of the last expression can be seen to cancel with the terms of $V^0\chi$ in the sum of (34*b*) and (35*b*). In symmetrizing (35*b*) with respect to permutations of $k_1, \ldots, k_n$, the factors $\exp(\mathrm{i}k_l)$ in the second part of (35*b*) can be replaced by $\tfrac{1}{2}\{\exp(\mathrm{i}k_l)+\exp(\mathrm{i}k_{l+1})\}$. One obtains accordingly

$$(T+V^0+V-E)\sum_{P_k}\chi_{k_1\ldots k_n} = -\tfrac{1}{4}\sum_{P_k}\sum_{l=1}^{n-1}[1+\exp\{-\mathrm{i}(K_l+K_{l+1})\}-2\exp(-\mathrm{i}K_l)]\{\exp(\mathrm{i}k_l)+\exp(\mathrm{i}k_{l+1})\} \times\exp(-\mathrm{i}X_{l-1})\chi^{(l)}_{k_1\ldots k_n}+\ldots. \tag{36}$$

With the right choice of the phase factors the function $\phi_{k_1\ldots k_n}$ given by (9*a*) can be seen to satisfy the wave equation (6*a*). The contribution from terms given explicitly in (36) vanishes because of the relationships (23*d*), the contribution from terms indicated by dots because of the relationships (18).

This concludes the verification that the new form of the wavefunctions given by $\phi_{k_1\ldots k_n}$ represents eigenfunctions of the Heisenberg Hamiltonian. Neither the form of the solution, nor the method of verification looks particularly one dimensional. The eminent initiator of this subject concluded his pioneering paper in 1931 with the remark that in a following work his methods would be extended to three-dimensional lattices, and the physical consequences concerning cohesion, ferromagnetism and conductivity would be drawn. The present work was started with the bold program of finding new ways to extend the one-dimensional solution to two and three dimensions, and work is in progress on this problem. The authors would not like to commit themselves, however, concerning the date of publication of the exact two and three dimensional solutions.

Appendix 1

Some of the identities and relationships referred to in the text simplify with the abbreviated notation

$$\alpha(X) = \frac{1}{1-\exp(\mathrm{i}X)}. \qquad \text{(A.1}a\text{)}$$

The relationship (33d) corresponds to the identity

$$\prod_{m=1}^{n} \alpha(X_m) = \sum_{m=1}^{n} \alpha(X_m) \prod_{l \neq m}^{n} \alpha(X_l - X_m). \qquad \text{(A.1}b\text{)}$$

This is trivially true for $n = 1$. For $n = 2$ it reads

$$\alpha(X_1)\alpha(X_2) = \alpha(X_1)\alpha(X_2 - X_1) + \alpha(X_2)\alpha(X_1 - X_2) \qquad \text{(A.1}c\text{)}$$

as it can be checked directly and has already been referred to in connection with the decompositions (14e,f) and (30d). If the identity is assumed to be valid for n, by applying first (A.1c), one obtains for $n+1$

$$\prod_{m=1}^{n+1} \alpha(X_m) = \alpha(X_{n+1}) \sum_{m=1}^{n} \alpha(X_m) \prod_{l \neq m}^{n} \alpha(X_l - X_m)$$

$$= \sum_{m=1}^{n} (\alpha(X_{n+1})\alpha(X_m - X_{n+1}) + \alpha(X_m)\alpha(X_{n+1} - X_m)) \prod_{l \neq m}^{n} \alpha(X_l - X_m)$$

$$= \alpha(X_{n+1}) \prod_{l=1}^{n} \alpha(X_l - X_{n+1}) + \sum_{m=1}^{n} \alpha(X_m) \prod_{l \neq m}^{n+1} \alpha(X_l - X_m) = \sum_{m=1}^{n+1} \alpha(X_m) \prod_{l \neq m}^{n+1} \alpha(X_l - X_m)$$

$$\text{(A.1}d\text{)}$$

which proves the identity by induction.

The identity

$$\sum_{m=1}^{n} \exp(-\mathrm{i}X_m) \prod_{l \neq m}^{n} \alpha(X_l - X_m) = 0 \qquad \text{(A.2}a\text{)}$$

for $n \geqslant 2$ which has been referred to in connection with the expression (33e) follows from (A.1b). The case $n = 2$ is seen to be valid because of

$$\exp(-\mathrm{i}X_1)\alpha(X_2 - X_1) = \frac{1}{\exp(\mathrm{i}X_1) - \exp(\mathrm{i}X_2)} = -\exp(-\mathrm{i}X_2)\alpha(X_1 - X_2). \qquad \text{(A.2}b\text{)}$$

If, with the help of the identity (A.1b), the last term of (A.2a) is rewritten as

$$\exp(-\mathrm{i}X_n) \prod_{l=1}^{n-1} \alpha(X_l - X_n) = \exp(-\mathrm{i}X_n) \sum_{m=1}^{n-1} \alpha(X_m - X_n) \prod_{l \neq m}^{n-1} \alpha(X_l - X_m) \qquad \text{(A.2}c\text{)}$$

the full sum in (A.2a) becomes

$$\sum_{m=1}^{n} \exp(-\mathrm{i}X_m) \prod_{l\neq m}^{n} \alpha(X_l - X_m)$$

$$= \sum_{m=1}^{n-1} \prod_{l\neq m}^{n-1} \alpha(X_l - X_m)\{\exp(-\mathrm{i}X_m)\alpha(X_n - X_m) + \exp(-\mathrm{i}X_n)\alpha(X_m - X_n)\} \quad \text{(A.2}d\text{)}$$

in which each term vanishes because of (A.2b).

Appendix 2

In order to show that the relationships (18) follow from the conditions (7), and that the contributions from the interaction terms which were indicated by dots in the text vanish, one has to consider $\tilde{\chi}_{k_1\ldots k_l\ldots k_m\ldots k_n}$ symmetrized with respect to the n cyclic permutations of $k_1, k_2, \ldots, k_n$.

For fixed l, m assume $l < m$. For the l cyclic permutations in which k_l stands before k_m and $r \leqslant l$, one has

$$\tilde{\chi}_{k_r\ldots k_l\ldots k_m\ldots k_{r-1}} = \prod_{\tilde{m}=1}^{l-r} \alpha(X_{\tilde{m};r}) \prod_{\tilde{m}=l-r+1}^{m-r} \alpha(X_{\tilde{m};r}) \prod_{\tilde{m}=m-r+1}^{n-1} \alpha(X_{\tilde{m};r}) \quad \text{(A.3}a\text{)}$$

where the notation

$$X_{\tilde{m};r} = K_1 + K_2 + \ldots + K_{\tilde{m}} - k_r - k_{r+1} - \ldots - k_{r+\tilde{m}-1} \quad \text{(A.3}b\text{)}$$

has been introduced. For $r = 1$, this factorization has been considered explicitly in (33a). For the $(n-m)$ similar cyclic permutations for which $r > m$, one has to replace r by $r-n$ on the product signs in (A.3a) and introduce the convention that in (A.3b) $k_{m'}$ is to be replaced by $k_{m'-n}$ for $m' > n$. For the $(m-l)$ cyclic permutations of $k_1, \ldots, k_n$ for which k_m precedes k_l one can write in a similar way

$$\tilde{\chi}_{k_{r'}\ldots k_m\ldots k_l\ldots k_{r'-1}} = \prod_{m'=1}^{m-r'} \alpha(X_{m';r'}) \prod_{m'=m-r'+1}^{l-r'+n} \alpha(X_{m';r'}) \prod_{m'=l-r'+1+n}^{n-1} \alpha(X_{m';r'}). \quad \text{(A.3}c\text{)}$$

The middle product in (A.3a) can be decomposed with the help of (A.1b), and for the sum of the $l+(n-m)$ cyclic permutations of this type one can write

$$\left(\sum_{r=1}^{l} + \sum_{r=m+1}^{n}\right)\tilde{\chi}_{k_r\ldots k_l\ldots k_m\ldots k_{r-1}}$$

$$= \sum_{r=1}^{l} \sum_{s=l-r+1}^{m-r} \alpha(X_{s;r})\left(\prod_{\tilde{m}=1}^{l-r} \alpha(X_{\tilde{m};r}) \prod_{\tilde{m}=l-r+1}^{s-1} \alpha(X_{\tilde{m};r} - X_{s;r}) \prod_{\tilde{m}=s+1}^{m-r} \alpha(X_{\tilde{m};r} - X_{s;r})\right.$$

$$\left.\times \prod_{\tilde{m}=m-r+1}^{n-1} \alpha(X_{\tilde{m};r})\right) + \sum_{r=m+1}^{n} \sum_{s=l-r+1+n}^{m-r+n} \alpha(X_{s;r})\left(\cdots\right)_{l-r+n,m-r+n} \quad \text{(A.4}a\text{)}$$

where the second large parenthesis is obtained from the first by adding n to $l-r$ and to $m-r$ on the product signs. In the similar decomposition of the middle product of (A.3c) one can separate the resulting sum into two parts, and for the sum of the related $(m-l)$

cyclic permutations one obtains

$$\sum_{r'=l+1}^{m} \tilde{\chi}_{k_{r'}\ldots k_m \ldots k_l \ldots k_{r'-1}}$$

$$= \sum_{r'=l+1}^{m} \left(\sum_{s'=m-r'+1}^{-r'+n} + \sum_{s'=-r'+n+1}^{l-r'+n} \right) \alpha(X_{s';r'}) \left\{ \prod_{m'=1}^{m-r'} \alpha(X_{m';r'}) \prod_{m'=m-r'+1}^{s'-1} \alpha(X_{m';r'} - X_{s';r'}) \right.$$

$$\left. \times \prod_{m'=s'+1}^{l-r'+n} \alpha(X_{m';r'} - X_{s';r'}) \prod_{m'=l-r'+1+n}^{n-1} \alpha(X_{m';r'}) \right\}. \qquad \text{(A.4}b\text{)}$$

One can establish a simple correspondence between the terms of the two pairs of double sums in (A.4*a*) and (A.4*b*). The $(m-l)l$ terms of the second double sum of (A.4*b*) occupy a domain in the $r's'$ plane bounded by the straight lines $r' = l+1$, $r' = m$, $r'+s' = 1+n$, $r'+s' = l+n$. By a linear change of variables

$$r' = r+s, \qquad s' = n-s \qquad \text{(A.4}c\text{)}$$

this goes over into a parallelogram of the rs plane bounded by the lines $r+s = l+1$, $r+s = m$, $r = 1$, $r = l$. This is the same domain of the rs plane as that occupied by the $l(m-l)$ terms of the first double sum of (4*a*). One can show that the products in braces in the individual terms of the two double sums as related to each other by the relationship (A.4*c*) differ for $k_1 + \ldots + k_n = K_1 + \ldots + K_n$ only through a cyclic permutation of $K_1, \ldots, K_n$. If one performs simultaneously with (A.4*c*) a cyclic permutation which depends on s,

$$K_1 \to K_{s+1}, K_2 \to K_{s+2}, \ldots, K_n \to K_{s+n}, \qquad \text{(A.4}d\text{)}$$

where the convention $K_{m'+n} = K_{m'}$ is used, the corresponding products transform exactly into each other.

With (A.3*b*), one has

$$X_{s+m';r} - X_{s;r} = K_{s+1} + K_{s+2} + \ldots + K_{s+m'} - k_{r+s} - k_{r+s+1} - \ldots - k_{r+s+m'-1} \qquad \text{(A.5}a\text{)}$$

which differs from $X_{m';r+s}$ only through the inverse of the cyclic permutation (A.4*d*). Through the simultaneous transformation (A.4*c*, *d*) one obtains therefore

$$X_{m';r'} \to X_{s+m';r} - X_{s;r}. \qquad \text{(A.5}b\text{)}$$

From the definitions, one has for $k_1 + \ldots + k_n = K_1 + \ldots + K_n$,

$$X_{n;r} = 0 \qquad \text{(A.5}c\text{)}$$

for all r, and the related convention

$$X_{m'+n;r} = X_{m';r} \qquad \text{(A.5}d\text{)}$$

can also be introduced. Since from (A.5*c*) one has $X_{s+(n-s);r} = 0$, with (A.5*b*) the transformations (A.4*c*, *d*) give

$$X_{s';r'} \to -X_{s;r}. \qquad \text{(A.5}e\text{)}$$

The difference between the transformations (A.5*b*) and (A.5*e*) reads

$$X_{m';r'} - X_{s';r'} \to X_{s+m';r}. \qquad \text{(A.5}f\text{)}$$

With (A.5*b*) the first product in the braces in (A.4*b*) transforms into the third product in the first term of (A.4*a*), with a simultaneous relabelling

$$\tilde{m} = s+m'. \tag{A.5g}$$

With the same relabelling, the second product in the braces in (A.4*b*) transforms into the fourth product in (A.4*a*). With a change

$$\tilde{m} = -(n-s)+m' \tag{A.5h}$$

and use of the convention (A.5*d*), the third product in (A.4*b*) goes over into the first product in (A.4*a*), and the fourth product in (A.4*b*) into the second product in (A.4*a*). The factor outside the brace in (A.4*b*) changes according to (A.5*e*) into $\alpha(-X_{s;r})$. A similar correspondence can be established between the $(m-l)(n-m)$ terms of the first double sum of (A.4*b*) and the $(n-m)(m-l)$ terms of the second double sum of (A.4*a*).

The sum of (A.4*a*, *b*) enters into the relationship (7) with additional phase factors and is then summed over all permutations of $K_1, \ldots, K_n$. If applied at the same time to the phase factors, the cyclic permutation (A.4*d*) therefore only interchanges two terms in this sum. With the notation (9*e*), one has under such a cyclic permutation

$$\theta_{\bar{l}\bar{m} \to s+\bar{l}, s+\bar{m}}. \tag{A.6a}$$

The phase factor shown explicitly in (9*a*) contains a sum of $\theta_{\bar{l}\bar{m}}$ for all pairs $\bar{l} < \bar{m}$ with $\bar{l}, \bar{m} = 1, 2, \ldots, n$. For some pairs $\bar{l}, \bar{m}$ the transformation (A.6*a*) leads to new indices $\tilde{l}, \tilde{m}$ in terms of the numbers $1, 2, \ldots, n$ for which $\tilde{l} < \tilde{m}$ and to a term $\theta_{\tilde{l}\tilde{m}}$ which was present in the original sum. For other pairs $\bar{l}$, $\bar{m}$ one will have $\tilde{l} > \tilde{m}$ and the corresponding term $\theta_{\tilde{l}\tilde{m}}$ will differ through a minus sign from a previous one, because of $\theta_{\tilde{l}\tilde{m}} = -\theta_{\tilde{m}\tilde{l}}$. The first case, $\tilde{l} < \tilde{m}$, results for values of $\bar{l}, \bar{m}$ for which $1 \leqslant \bar{l} < \bar{m} \leqslant n-s$ or

$$n-s < \bar{l} < \bar{m} \leqslant n$$

and the new indices $\tilde{l}$, $\tilde{m}$ will be in the intervals $s < \tilde{l} < \tilde{m} \leqslant n$ or $1 \leqslant \tilde{l} < \tilde{m} \leqslant s$. The second case, $\tilde{l} > \tilde{m}$, results for $1 \leqslant \bar{l} \leqslant n-s$ and $n-s < \bar{m} \leqslant n$ for which $s < \tilde{l} \leqslant n$, $1 \leqslant \tilde{m} \leqslant s$. The total phase factor is multiplied by a factor effecting the related changes of sign and the cyclic permutation (A.4*d*) leads to

$$\exp\left(-\tfrac{1}{2}\mathrm{i}\sum_{\bar{l}<\bar{m}} \theta_{\bar{l}\bar{m}}\right) \to \exp\left(-\tfrac{1}{2}\mathrm{i}\sum_{\bar{l}<\bar{m}} \theta_{\bar{l}\bar{m}}\right) \exp\left(\mathrm{i}\sum_{\bar{l}=1}^{s}\sum_{\bar{m}=s+1}^{n} \theta_{\bar{l}\bar{m}}\right). \tag{A.6b}$$

With the previously established relationships between the terms of (A.4*a*, *b*), for $k_1+\ldots+k_n = K_1+\ldots+K_n$ one can write accordingly

$$\sum_{P_K(\text{cycl})} \exp\left(-\tfrac{1}{2}\mathrm{i}\sum_{\bar{l}<\bar{m}} \theta_{\bar{l}\bar{m}}\right) \sum_{P_k(\text{cycl})} \tilde{\chi}_{k_1\ldots k_l\ldots k_m\ldots k_n}$$
$$= \sum_{P_K(\text{cycl})} \exp\left(-\tfrac{1}{2}\mathrm{i}\sum_{\bar{l}<\bar{m}} \theta_{\bar{l}\bar{m}}\right)\left(\sum_{r=1}^{l}\sum_{s=l-r+1}^{m-r} + \sum_{r=m+1}^{n}\sum_{s=l-r+1+n}^{m-r+n}\right)\mathscr{P}_{rs} \tag{A.7a}$$

with

$$\mathscr{P}_{rs} = \left(\prod_{\tilde{m}=1}^{l-r} \alpha(X_{\tilde{m};r}) \prod_{\substack{\tilde{m}=l-r+1\\ \tilde{m}\neq s}}^{m-r} \alpha(X_{\tilde{m};r}-X_{s;r}) \prod_{\tilde{m}=m-r+1}^{n-1} \alpha(X_{\tilde{m};r})\right)$$
$$\times\left\{\alpha(X_{s;r})+\alpha(-X_{s;r})\exp\left(\mathrm{i}\sum_{\bar{l}=1}^{s}\sum_{\bar{m}=s+1}^{n} \theta_{\bar{l}\bar{m}}\right)\right\}. \tag{A.7b}$$

The additional convention is applied that in the second double sum of (A.7a) n is to be added to $l-r$ and to $m-r$ on the product signs in $\mathscr{P}_{rs}$. This expression is now in such a form that, if one replaces k_l by k and k_m by k_l+k_m-k, the only dependence on k is through $\alpha(X_{s;r})$ and $\alpha(-X_{s;r})$ in the last brace of $\mathscr{P}_{rs}$. If one sums with respect to k, according to (11a), $\alpha(X_{s;r})$ is replaced by $\alpha(NK_1+\ldots+NK_s)$ and $\alpha(-X_{s;r})$ by $\alpha(-NK_1-\ldots-NK_s)$. From the definition (A.1a) of $\alpha(X)$, one can write

$$\alpha(-NK_1-\ldots-NK_s) = -\alpha(NK_1+\ldots+NK_s)\exp \mathrm{i}(NK_1+\ldots+NK_s) \tag{A.7c}$$

and the effect of the summation over k on (A.7a, b) can be seen to be a replacement of the last parenthesis in (A.7b) by

$$\alpha(NK_1+\ldots+NK_s)\left[1-\exp\left\{\mathrm{i}\left(NK_1+\ldots+NK_s+\sum_{\bar{l}=1}^{s}\sum_{\bar{m}=s+1}^{n}\theta_{\bar{l}\bar{m}}\right)\right\}\right]. \tag{A.7d}$$

If the n equations (18) are satisfied, then the n expressions (A.7d) with $s = 1, 2, \ldots, n$ vanish. The equation

$$\exp\left\{\mathrm{i}\left(NK_1+\ldots+NK_s+\sum_{\bar{l}=1}^{s}\sum_{\bar{m}=s+1}^{n}\theta_{\bar{l}\bar{m}}\right)\right\} = 1 \tag{A.7e}$$

is for $s = 1$ identical with (18) for $\bar{l} = 1$. If one assumes (A.7e) and

$$\exp\left(\mathrm{i}NK_{s+1}+\sum_{\bar{m}=1}^{n}\theta_{s+1,\bar{m}}\right) = 1 \tag{A.7f}$$

the product of the last two equations establishes (A.7e) for $s+1$. Conversely from (A.7e) for $s = 1, 2, \ldots, n$ follow the equations (18).

In order to show that these equations follow from the conditions (7), one has to substitute in the latter the wavefunction (9a). This results from (A.7a) by adding the $(n-1)!$ permutations with respect to $k_1, \ldots, k_n$ which are not included in the chosen n cyclic permutations, and then performing a similar symmetrization with respect to $(n-1)!$ permutations of $K_1, \ldots, K_n$. In order that the resulting equations (7) should be valid for arbitrary values of $k_1, \ldots, k_n$ the coefficients (A.7d) have to vanish. The equations (A.7e), and their permutations with respect to $K_1, \ldots, K_n$ follow. As a result of (A.7e) it follows on the other hand, that in replacing the full wavefunctions by their cyclic parts (A.7a) in the equations (7), the corresponding sums also vanish.

Almost exactly the same argument leads to the conclusion that the sum of contributions to the interaction term $V^0\phi$ indicated by dots in (33c) and (34b) vanishes. After replacing k_l by k and k_m by k_l+k_m-k and before performing the summation over k, the cyclic sum (A.7a) is still to be multiplied by ϵ_{k,k_l+k_m-k} in this case. The dependence on k is now in the product of the last brace of (A.7b) and of this factor. According to the relationship (25d), the contributions indicated by dots will contain this brace with $\alpha(X_{r;s})$ replaced by $\alpha(NK_1+\ldots+NK_s)$, and $\alpha(-X_{r;s})$ by $\alpha(-NK_1-\ldots-NK_s)$ as before, multiplied by the common factor $\epsilon_{k_l+X_{s;r},k_m-X_{s;r}}$. The vanishing of (A.7$d$) results therefore in the vanishing of these contributions. The vanishing of related contributions to $V\phi$ follows as a consequence of the identity (8a).

Appendix 3

The relationships (23d) can be derived from the zero-momentum condition (21e) as follows. In order to separate some of the dependence of the wavefunction (9a, b, c, d)

on k_1, consider the n permutations $\tilde{\chi}_{k_2\ldots k_l k_1 k_{l+1}\ldots k_n}$ of $\tilde{\chi}_{k_1\ldots k_n}$ in which k_1 is on the lth place. With the notation (A.1*a*), for $1 < l < n-1$ one can write

$$\tilde{\chi}_{k_2\ldots k_l k_1 k_{l+1}\ldots k_n} = \left(\prod_{m=1}^{l-2} \alpha(X_{m;2})\right)\alpha(X_{l-1;2})\alpha(X_l)\left(\prod_{m=l+1}^{n-1} \alpha(X_m)\right) \tag{A.8a}$$

where, as before

$$X_m = X_{m;1} = K_1+\ldots+K_m-k_1-\ldots-k_m \tag{A.8b}$$

$$X_{m;2} = K_1+\ldots+K_m-k_2-\ldots-k_{m+1}. \tag{A.8c}$$

From (A.1*c*), the two middle factors in (A.8*a*) can be written as

$$\alpha(X_{l-1;2})\alpha(X_l) = \alpha(X_l-X_{l-1;2})\alpha(X_{l-1;2})+\alpha(X_{l-1;2}-X_l)\alpha(X_l) \tag{A.9a}$$

in which one can substitute

$$X_l-X_{l-1;2} = K_l-k_1. \tag{A.9b}$$

For $l = 1$ and $l = n$, one has

$$\tilde{\chi}_{k_1\ldots k_n} = \alpha(K_1-k_1)\prod_{m=2}^{n-1} \alpha(X_m) \tag{A.9c}$$

and if $k_1+\ldots+k_n = K_1+\ldots+K_n$,

$$\tilde{\chi}_{k_2\ldots k_n k_1} = \left(\prod_{m=1}^{n-2} \alpha(X_{m;2})\right)\alpha(k_1-K_n). \tag{A.9d}$$

Substituting (A.9*a*, *b*) into (A.8*a*) for $1 < l < n-1$, and summing over l, one obtains

$$\sum_{l=1}^{n} \tilde{\chi}_{k_2\ldots k_l k_1 k_{l+1}\ldots k_n} = \sum_{l=1}^{n-1} (\alpha(K_l-k_1)+\alpha(k_1-K_{l+1}))\left(\prod_{m=1}^{l-1} \alpha(X_{m;2})\right)\left(\prod_{m=l+1}^{n-1} \alpha(X_m)\right) \tag{A.9e}$$

where the first of the last two factors is to be replaced by unity for $l = 1$, and the second similarly for $l = n-1$.

The summation P_k over the $n!$ permutations of $k_1, \ldots, k_n$ in the wavefunction (9*a*) can be considered as a summation $P_k^{(n-1)}$ of the $(n-1)!$ permutations of $k_2, \ldots, k_n$ of the n terms of the left hand side of (A.9*e*). For $k_1 = 0$, the wavefunction (9*a–d*) can be written accordingly as

$$\phi_{k_1=0,k_2\ldots k_n} = \sum_{P_K} \exp\left(-\tfrac{1}{2}\mathrm{i}\sum_{l<\bar{m}} \theta_{l\bar{m}}\right)\sum_{l=1}^{n-1}\left(\frac{1}{1-\exp(\mathrm{i}K_l)}+\frac{1}{1-\exp(-\mathrm{i}K_{l+1})}\right)$$
$$\times \sum_{P_k^{(n-1)}}\left(\prod_{m=1}^{l-1} \alpha(X_{m;2})\prod_{m=l+1}^{n-1} \alpha(X_m)\right)_{k_1=0}. \tag{A.10}$$

For a given l, the last two products in (A.10) contain K_l or K_{l+1} only through the sum K_l+K_{l+1}, and are therefore invariant with respect to an interchange of K_l and K_{l+1}. Since (A.10) contains a summation over all permutations of $K_1, \ldots, K_n$, factors of the form of the left hand side of equation (22*b*) in which K_1, K_2 are replaced by K_l, K_{l+1} or by any two values K_l, K_m selected from $K_1, \ldots, K_n$, can be separated from its terms. If all these factors vanish, that is, if equations (23*d*) are satisfied for $l \neq m = 1, 2, \ldots, n$, then (A.10) vanishes and one has $\phi_{k_1=0,k_2\ldots k_n} = 0$. Conversely, if this equation is valid for all $(N)^{n-1}$ sets of wavenumbers $k_2, \ldots, k_n$, one can look at these $(N)^{n-1}$ equations as linear homogeneous equations for the $\binom{n}{2}$ factors of the type of the left hand side of equation (22b), though the coefficients of these equations also depend on $K_1, \ldots, K_n$. The interested reader may try to convince himself that in general it will be possible to select $\binom{n}{2}$ linearly independent equations from these $(N)^{n-1}$ linear homogeneous equations, and these will consequently have the unique solution in which all the unknowns, that is the $\binom{n}{2}$ factors of the type mentioned, vanish. Equations (23d) will then follow from the conditions $\phi_{k_1=0,k_2\ldots k_n} = 0$.

Appendix 4

The identity (35a) will be proved by induction in the equivalent form which, for $(n+1)$ and for $k_1 + \ldots + k_{n+1} = K_1 + \ldots + K_{n+1}$, is given by

$$\epsilon_{K_1\ldots K_{n+1}} - \epsilon_{k_1\ldots k_{n+1}} = \tfrac{1}{2}\sum_{l=1}^{n} \{-\exp(-\mathrm{i}k_l) - \exp(\mathrm{i}k_{l+1}) + \exp(\mathrm{i}K_l) + \exp(-\mathrm{i}K_{l+1})\} \times \{1 - \exp(-\mathrm{i}X_l)\}. \quad (\mathrm{A.11}a)$$

For $n = 2$, the identity (35a) is valid in the form (27a). Assume the relationship (A.11a) to be valid for n, and with the notation

$$\tilde{K}_n = k_1 + \ldots + k_n - K_1 - \ldots - K_{n-1} \quad (\mathrm{A.11}b)$$

write

$$\epsilon_{K_1\ldots \tilde{K}_n} - \epsilon_{k_1\ldots k_n}$$
$$= \tfrac{1}{2}\sum_{l=1}^{n-2} \{-\exp(-\mathrm{i}k_l) - \exp(\mathrm{i}k_{l+1}) + \exp(\mathrm{i}K_l) + \exp(-\mathrm{i}K_{l+1})\}\{1 - \exp(-\mathrm{i}X_l)\}$$
$$+ \tfrac{1}{2}\{-\exp(-\mathrm{i}k_{n-1}) - \exp(\mathrm{i}k_n) + \exp(\mathrm{i}K_{n-1}) + \exp(-\mathrm{i}\tilde{K}_n)\}\{1 - \exp(-\mathrm{i}X_{n-1})\}. \quad (\mathrm{A.11}c)$$

One has

$$\epsilon_{K_1\ldots K_{n+1}} - \epsilon_{k_1\ldots k_{n+1}} = (\epsilon_{K_1\ldots \tilde{K}_n} - \epsilon_{k_1\ldots k_n}) + (\epsilon_{K_n K_{n+1}} - \epsilon_{\tilde{K}_n k_{n+1}}). \quad (\mathrm{A.11}d)$$

Since, for $k_1 + \ldots + k_{n+1} = K_1 + \ldots + K_{n+1}$, one has

$$K_n + K_{n+1} = K_n + (k_1 + \ldots + k_{n+1} - K_1 - \ldots - K_n) = \tilde{K}_n + k_{n+1} \quad (\mathrm{A.12}a)$$

for the part $(\epsilon_{K_n K_{n+1}} - \epsilon_{\tilde{K}_n k_{n+1}})$ of (A.11d) one can apply the identity for $n = 2$. Noting that

$$K_n - \tilde{K}_n = X_n, \quad (\mathrm{A.12}b)$$

one can write accordingly

$$\epsilon_{K_n K_{n+1}} - \epsilon_{\tilde{K}_n k_{n+1}} = \tfrac{1}{2}\{-\exp(-\mathrm{i}\tilde{K}_n) - \exp(\mathrm{i}k_{n+1}) + \exp(\mathrm{i}K_n) + \exp(-\mathrm{i}K_{n+1})\} \times \{1 - \exp(-\mathrm{i}X_n)\}. \quad (\mathrm{A.12}c)$$

The sum of (A.11c) and (A.12c), substituted into (A.11d), gives (A.11a), with the help of the identity

$$\{\exp(-\mathrm{i}\tilde{K}_n) - \exp(-\mathrm{i}K_n)\}\{1 - \exp(-\mathrm{i}X_{n-1})\} = \{\exp(-\mathrm{i}\tilde{K}_n) - \exp(-\mathrm{i}k_n)\} \times \{1 - \exp(-\mathrm{i}X_n)\}. \quad (\mathrm{A.12}d)$$

This last relationship can be checked by performing the multiplications, and comparing the two sides of the equation term by term, making use of (A.12*a*, *b*) and the definitions of X_{n-1}, X_n. The relationship (A.11*a*) given for $(n+1)$ follows therefore from its validity for $n = 2$ and from the assumption that it is valid for n.

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